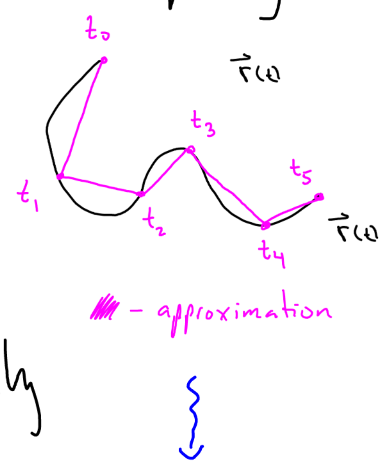
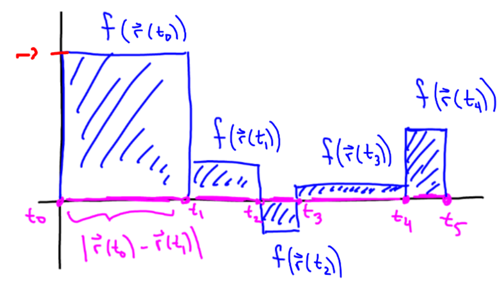
Line Integrals

Given a space curve $\vec{r} : I \to \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$, one might like to compute the accumulated values of f along $\vec{r}(t)$; this naturally amounts to computing an integral. We approximate the curve $\vec{r}(t)$ by line segments piecewise and add the values at the left endpoints scaled for the length of the segment. In doing so, we essentially "unfold" the curve and create rectangles with height determined by the function and base length the length of the segment.





The corresponding approximation is given by the formula

$$A_n = \sum_{k=1}^{n} f(\vec{r}(t_{k-1})) \left| \frac{\vec{r}(t_k) - \vec{r}(t_{k-1})}{(\Delta t_k)^2} \right| \Delta t_k .$$

Limiting approximations of the curve we obtain the formula

$$\int_C f \, ds := \lim_{n \to \infty} \sum_{k=1}^{n} f(\vec{r}(t_{k-1})) \left| \frac{\vec{r}(t_k) - \vec{r}(t_{k-1})}{(\Delta t_k)^2} \right| \Delta t_k$$

$$= \int_{t=a}^{b} f(\vec{r}(t)) \, |\vec{r}\,'(t)| \, dt$$

NB: Line integrals detect orientation changes! $\int_{-C} f \, ds = -\int_C f \, ds$

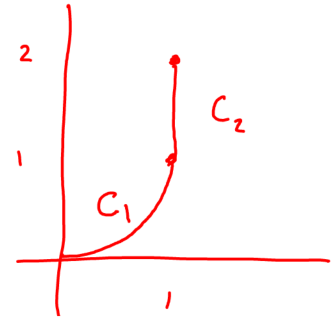
Ex: Compute $\int_C 2x\, ds$ for C the arc of $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the segment from $(1,1)$ to $(1,2)$.

Sol:

$$\int_C f\, ds = \int_{t=a}^{b} f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$$

$$\| $$

$$\int_{C_1} f\, ds + \int_{C_2} f\, ds$$



$r_1(t) = \langle t, t^2 \rangle$ for $0 \le t \le 1$ $\qquad$ $\vec{r}_2(t) = \langle 1, t \rangle$ for $1 \le t \le 2$

$\vec{r}_1\,'(t) = \langle 1, 2t \rangle$ $\qquad$ $\vec{r}_2\,'(t) = \langle 0, 1 \rangle$

$|\vec{r}_1\,'(t)| = \sqrt{1 + 4t^2}$ $\qquad$ $|\vec{r}_2\,'(t)| = \sqrt{0^2 + 1^2} = 1$

$f(\vec{r}_1(t)) = 2 \cdot t = 2t$ $\qquad$ $f(\vec{r}_2(t)) = 2 \cdot 1 = 2$

$$\therefore \int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds$$

$$= \int_{t=0}^{1} 2t\sqrt{1+4t^2}\, dt + \int_{t=1}^{2} 2 \cdot 1\, dt$$

$\left(\begin{array}{ll} u = 1+4t^2 & u(1) = 5 \\ du = 8t\, dt & u(0) = 1 \end{array} \right)$

$$= \int_{t=0}^{1} \frac{1}{4} (1+4t^2)^{1/2}\, 8t\, dt + 2\left[t \right]_{t=1}^{2}$$

$$= \frac{1}{4}\int_{u=1}^{5} u^{1/2}\, du + 2(2-1) = \frac{1}{4} \cdot \frac{2}{3} \left[u^{3/2} \right]_{u=1}^{5} + 2$$

$$= \frac{1}{6}\left(5^{3/2} - 1 \right) + 2$$

$\boxed{/\!/}$

Ex: Compute $\int_C (2 + x^2 y)\, ds$ for C the upper-half of the unit circle (parametrized counter-clockwise).

Sol:

$$\int_C f\, ds = \int_{t=a}^{b} f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$$

$$= \int_{t=0}^{\pi} \left(2 + \cos^2(t)\sin(t)\right) \cdot 1\, dt$$

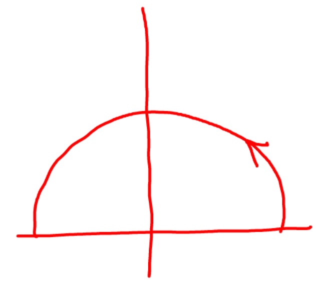
$$= \int_{t=0}^{\pi} 2\, dt + \int_{t=0}^{\pi} \cos^2(t)\sin(t)\, dt$$

$$\begin{cases} u = \cos(t) & u(0) = 1 \\ du = -\sin(t)\,dt & u(\pi) = -1 \end{cases}$$

$$= 2\left[t\right]_{t=0}^{\pi} - \int_{u=1}^{-1} u^2\, du$$

$$= 2(\pi - 0) - \frac{1}{3}\left[u^3\right]_{u=1}^{-1}$$

$$= 2\pi - \frac{1}{3}(-1 - 1) = 2\pi + \frac{2}{3} = \frac{6\pi + 2}{3}$$



$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$

$0 \le t \le \pi$

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$

$|\vec{r}\,'(t)| = \sqrt{\sin^2(t) + \cos^2(t)}$

$= 1$

☐

Ex: Compute $\int_C xy \sin(z)\, ds$ for $C(t) = \langle \cos(t), \sin(t), t \rangle$ with $0 \le t \le \frac{7\pi}{4}$.

Sol: Practice exercise for you :)

☐

Defn: For $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ and curve $c : [a,b] \to \mathbb{R}^n$ we define $\int_C f \, dx_k := \int_{t=a}^{b} f(c(t)) \, x_k'(t) \, dt$ where

$$c(t) = \langle x_1(t), x_2(t), \cdots, x_n(t) \rangle.$$

NB: It's customary to abbreviate $\int_C P\,dx + Q\,dy := \int_C P\,dx + \int_C Q\,dy$.

Ex: Compute $\int_C y^2\,dx + x\,dy$

for C the line segment from $(-5,-3)$ to $(0,2)$.

Sol: Recall: the line segment connecting $\overrightarrow{AB}$ is parametrized by $\vec{r}(t) = (1-t)\overrightarrow{OA} + t\,\overrightarrow{OB}$ for $0 \le t \le 1$.

$$\vec{r}(t) = (1-t)\langle -5,-3 \rangle + t\langle 0,2 \rangle$$

$$= \langle 5t-5,\ 5t-3 \rangle \qquad \text{So} \qquad \vec{r}\,'(t) = \langle 5,5 \rangle$$

$$\therefore \int_C y^2\,dx + x\,dy$$

$$= \int_{t=0}^{1} (5t-5)^2 5\,dt + \int_{t=0}^{1} (5t-3)5\,dt$$

$$= 5\int_{t=0}^{1} \left(25t^2 - 50t + 25 + 5t - 3\right) dt$$

$$= 5\int_{t=0}^{1} \left(25t^2 - 45t + 22\right) dt$$

$$= 5\left[\frac{25}{3}t^3 - \frac{45}{2}t^2 + 22t\right]_{t=0}^{1} = 5\left(\left(\frac{25}{3} - \frac{45}{2} + 22\right) - 0\right)$$

$$= \frac{5}{6}(50 - 135 + 132) = \frac{5 \cdot 47}{6} = \frac{235}{6}$$

$\blacksquare$

Ex: Compute $\int_C y^2\,dx + x\,dy$ for C the arc of $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Sol: $\vec{r}(t) = \langle 4 - t^2, t \rangle \rightsquigarrow \vec{r}\,'(t) = \langle -2t, 1 \rangle$

Bounds: $\vec{r}(t) = \langle -5, -3 \rangle \iff \begin{cases} 4 - t^2 = -5 \\ t = -3 \end{cases}$ ✓ $a = -3$

$\vec{r}(t) = \langle 0, 2 \rangle \iff \begin{cases} 4 - t^2 = 0 \\ t = 2 \end{cases}$ ✓ $b = 2$

$$\therefore \int_C y^2\,dx + x\,dy = \int_{t=-3}^{2} \left(t^2 \cdot -2t + (4 - t^2)\cdot 1 \right) dt$$

$$= \int_{t=-3}^{2} \left(-2t^3 - t^2 + 4 \right) dt$$

$$= \left[-\frac{1}{2} t^4 - \frac{1}{3} t^3 + 4t \right]_{t=-3}^{2}$$

$$= \left(-\frac{16}{2} - \frac{8}{3} + 8 \right) - \left(-\frac{81}{2} - \frac{-27}{3} - 12 \right)$$

$$= -\frac{8}{3} + \frac{81}{2} + 3$$

$$= \frac{-16 + 243 + 18}{6} = \frac{245}{6}$$

$\blacksquare$

Ex: Compute $\int_C y\,dx + z\,dy + x\,dz$ for C the line segments $(2,0,0)$ to $(3,4,5)$ followed by $(3,4,5)$ to $(3,4,0)$.

Sol: Let C_1 and C_2 be the line segments from $(2,0,0)$ to $(3,4,5)$ and $(3,4,5)$ to $(3,4,0)$ respectively.

$$\vec{r}_1(t) = t\langle 2,0,0\rangle + (1-t)\langle 3,4,5\rangle = \langle 3-t,\ 4-4t,\ 5-5t\rangle \quad 0 \le t \le 1$$

$$\vec{r}_1{}'(t) = \langle -1, -4, -5\rangle$$

$$\vec{r}_2(t) = t\langle 3,4,5\rangle + (1-t)\langle 3,4,0\rangle = \langle 3, 4, 5t\rangle \quad 0 \le t \le 1$$

$$r_2'(t) = \langle 0, 0, 5\rangle.$$

$$\int_C y\,dx + z\,dy + x\,dz = \int_{C_1} y\,dx + z\,dy + x\,dz + \int_{C_2} y\,dx + z\,dy + x\,dz$$

$$= \int_{t=0}^{1} \left((4-4t)\cdot -1 + (5-5t)\cdot -4 + (3-t)\cdot -5 \right) dt + \int_{t=0}^{1} (4\cdot 0 + 5t\cdot 0 + 3\cdot 5)\,dt$$

$$= \int_{t=0}^{1} (15 + 29t - 15 - 20 - 4)\,dt$$

$$= \int_{t=0}^{1} (29t - 24)\,dt$$

$$= \left[\frac{29}{2}t^2 - 24t \right]_{t=0}^{1}$$

$$= \frac{29}{2}(1^2 - 0^2) - 24(1-0)$$

$$= \frac{29}{2} - 24 = \frac{29-48}{2} = -\frac{18}{2}$$

We can also compute line integrals of vector fields along curves.

<u>Defn</u>: Given vector field $\vec{v} : \mathbb{R}^n \to \mathbb{R}^n$ and curve C parametrized by $\vec{r} : [a,b] \to \mathbb{R}^n$, the <u>line integral</u> of $\vec{v}$ along C is

$$\int_C \vec{v} \cdot d\vec{r} := \int_{t=a}^{b} \vec{v}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

<u>NB</u>: Line integrals are independent of (piecewise-smooth) parametrization.

We typically interpret the line integral of a vector field as work done by a force acting on a particle... ☆

<u>Ex</u>: Compute the work done by $\vec{V}(x,y) = \langle x^2, -xy \rangle$ acting on a particle moving along $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ w/ $0 \le t \le \frac{\pi}{4}$.

<u>Sol</u>: $\displaystyle\int_C \vec{v} \cdot d\vec{r} = \int_{t=a}^{b} \vec{v}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$$\vec{v}(\vec{r}(t)) = \langle \cos^2(t), -\cos(t)\sin(t) \rangle$$

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$$

$$\vec{v}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -\sin(t)\cos^2(t) + \cos(t) \cdot -\cos(t)\sin(t)$$

$$= -2\cos^2(t)\sin(t)$$

$$= \int_{t=0}^{\pi/4} 2\cos^2(t) \cdot -\sin(t)\, dt$$

$\left(\begin{array}{l} u = \cos(t) \\ du = -\sin(t)\, dt \end{array} \right.$ $\quad u(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$, $\quad u(0) = 1$

$$= 2\int_{u=1}^{1/\sqrt{2}} u^2\, du = \frac{2}{3}\left[u^3\right]_{u=1}^{1/\sqrt{2}} = \frac{2}{3}\left(\left(\frac{1}{\sqrt{2}}\right)^3 - 1\right) = \frac{2}{3}\left(\frac{1}{\sqrt{8}} - 1\right) \qquad \boxed{//}$$

Ex: Evaluate $\int_C \vec{v}\cdot d\vec{r}$ where $\vec{v}(x,y,z) = \langle xy, yz, zx \rangle$
where C is the moment curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ on $0 \le t \le 1$.

Sol: $\int_C \vec{v}\cdot d\vec{r} = \int_{t=a}^{b} \vec{v}(\vec{r}(t))\cdot \vec{r}\,'(t)\, dt$

$$\vec{v}(\vec{r}(t)) = \langle t^3, t^5, t^4 \rangle$$

$$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$\therefore\ \vec{v}(\vec{r}(t))\cdot\vec{r}\,'(t) = t^3 + 2t^6 + 3t^6 = t^3 + 5t^6$$

$$\therefore \int_C \vec{v}\cdot d\vec{r} = \int_{t=0}^{1} \left(t^3 + 5t^6\right) dt$$

$$= \left[\frac{1}{4}t^4 + \frac{5}{7}t^7\right]_{t=0}^{1}$$

$$= \left(\frac{1}{4} + \frac{5}{7}\right) - 0 = \frac{27}{28}$$

$\blacksquare$

Fundamental Theorem for Line Integrals

Line integrals are inherently one-dimensional, so it seems reasonable to expect they might satisfy an analogue of the Fundamental Theorem of Calculus... Here you go ☺

Prop (Fundamental Theorem of Line Integrals): Suppose C is a (piecewise-) smooth curve with parametrization $\vec{r}:[a,b]\to\mathbb{R}^n$. If $f:\mathbb{R}^n\to\mathbb{R}$ is differentiable along C and ∇f is continuous on C, then $\int_C \vec{\nabla} f\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$.

$\vec{r}(a)$ → $\vec{r}(b)$

pf: Given the above assumptions, we have:

$$\int_C \vec{\nabla} f\cdot d\vec{r} = \int_{t=a}^{b} \vec{\nabla} f(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

(definition)

$$= \int_{t=a}^{b} \frac{d}{dt}\left[f(\vec{r}(t))\right]dt$$

(★ multivariate chain rule) — single variable function!

$$= \left[f(\vec{r}(t))\right]_{t=a}^{b} = f(\vec{r}(b)) - f(\vec{r}(a))$$

(★ FTC) (evaluation) ▨

NB: This is really a statement about conservative vector fields: a potential function is essentially an antiderivative of a conservative vector field...

When $\vec{v}$ is conservative, the FToLI is often a much simpler and more efficient method of computing a line integral. ☆

Ex: Compute $\int_C \vec{v} \cdot d\vec{r}$ for $\vec{v}(x,y) = \langle 3 + 2xy^2, 2x^2y \rangle$ on the curve $\vec{r}(t) = \langle t, \frac{1}{t} \rangle$ for $1 \le t \le 4$.

Sol: ① Check $\vec{v}$ is conservative.

$$\frac{\partial v_x}{\partial y} = 4xy = \frac{\partial v_y}{\partial x} \quad \text{So } \vec{v} = \vec{\nabla} f \text{ is conservative}$$

② Compute a potential.

$$f(x,y) = \int (3 + 2xy^2)\, dx = 3x + x^2y^2 + C(y)$$

$$\therefore 2x^2y = \frac{\partial}{\partial y}\left[3x + x^2y^2 + C(y)\right] = 2x^2y + C'(y)$$

$$\therefore C'(y) = 0 \qquad \therefore C(y) = \int C'(y)\, dy = D$$

$\therefore f(x,y) = 3x + x^2y^2$ is a potential function for $\vec{v}$.

③ Apply FToLI: $\quad \vec{r}(t) = \langle t, \frac{1}{t} \rangle$, $a = 1$, $b = 4$

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(4, \tfrac{1}{4}) - f(1,1)$$

$$= \left(3 \cdot 4 + 4^2 \cdot (\tfrac{1}{4})^2\right) - \left(3 \cdot 1 + 1^2 \cdot (\tfrac{1}{1})^2\right)$$

$$= 12 + 1 - 3 - 1 = 9$$

//

Exercise: Try to compute $\int_C \vec{v} \cdot d\vec{r}$ without FTLI ...

Ex: Compute $\int_C \vec{v}\cdot d\vec{r}$ for $\vec{v}(x,y) = e^{xy}\langle 1+xy,\ x^2\rangle$

on the curve $\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$ for $0 \le t \le \frac{\pi}{2}$.

Sol: Part 1: (W/o FTLI):

$$\int_C \vec{v}\cdot d\vec{r} = \int_{t=a}^{b} \vec{v}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$\vec{v}(\vec{r}(t)) = e^{\cos(t)\sin(t)}\langle 1+\cos(t)\sin(t),\ \cos^2(t)\rangle$$

$$\vec{r}\,'(t) = \langle -\sin(t),\ \cos(t)\rangle$$

$$\therefore\ \vec{v}(\vec{r}(t))\cdot\vec{r}\,'(t) = e^{\cos(t)\sin(t)}\left(-\sin(t)\left(1+\cos(t)\sin(t)\right)+\cos^3(t)\right)$$

$$= e^{\cos(t)\sin(t)}\left(-\sin(t) - \cos(t)\sin^2(t) + \cos(t)\left(1-\sin^2(t)\right)\right)$$

$$= e^{\cos(t)\sin(t)}\left(\cos(t)-\sin(t) - 2\cos(t)\sin^2(t)\right)$$

→ NB: Integrating this would be awful...

$$\int_C \vec{v}\cdot d\vec{r} = \int_{t=0}^{\pi/2} e^{\cos(t)\sin(t)}\left(\cos(t)-\sin(t)-2\cos(t)\sin^2(t)\right)dt$$

→ Terrible... Don't know where to start...

Part 2: (With FTLI): $\vec{v} = e^{xy}\langle 1+xy,\ x^2\rangle$

① Check conservativity.

$$\frac{\partial v_x}{\partial y} = \frac{\partial}{\partial y}\left[e^{xy}(1+xy)\right] = e^{xy}(x) + (1+xy)\,x e^{xy} = x(2+xy)e^{xy}$$

$$\frac{\partial v_y}{\partial x} = \frac{\partial}{\partial x}\left[e^{xy}x^2\right] = e^{xy}\cdot 2x + y x^2 e^{xy} = x e^{xy}(2+xy)$$

(the two results are equal: //)

$$\therefore\ \vec{v} = \vec{\nabla} f \text{ for some function } f(x,y)$$

② Compute Potential:

$f(x,y) = \int x^2 e^{xy}\,dy = xe^{xy} + C(x)$

$\therefore (1+xy)e^{xy} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[xe^{xy} + C(x)\right]$

$= xye^{xy} + e^{xy} + C'(x)$

$= (1+xy)e^{xy} + C'(x)$

$\therefore C'(x) = 0 \qquad \therefore C(x) = D$

$\therefore f(x,y) = xe^{xy}$ is a potential for $\vec{v}$.

③ Use FTLI.

$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ on $0 \le t \le \frac{\pi}{2}$

$\int_C \vec{v}\cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$\begin{cases} \vec{r}(0) = \langle 1, 0 \rangle \\ \vec{r}(\frac{\pi}{2}) = \langle 0, 1 \rangle \end{cases}$

$= f(0,1) - f(1,0)$

$= 0\cdot e^{0\cdot 1} - 1\cdot e^{1\cdot 0} = -1$ ///

Ex: Compute $\int_C \vec{v}\cdot d\vec{r}$ for $\vec{v}(x,y,z) = \langle y^2 z + 2xz^2,\ 2xyz,\ xy^2 + 2x^2 z \rangle$ on $\vec{r}(t) = \langle \sqrt{t}, t+1, t^2 \rangle$ w/ $0 \le t \le 2$.

Sol: Left to you as an exercise ☺ ///

NB: $\vec{v}(x,y,z)$ has three variables, so you'll need to exercise some caution computing its potential function...

Ex: Compute $\int_C \vec{v} \cdot d\vec{r}$ for $\vec{v}(x,y,z) = \langle \sin(y), x\cos(y) + \cos(z), -y\sin(z) \rangle$

on $\vec{r}(t) = \langle \sin(t), t, 2t \rangle$ w/ $0 \le t \le \frac{\pi}{3}$.

Sol: Check conservativity:

$$\frac{\partial v_x}{\partial y} = \frac{\partial}{\partial y}[\sin(y)] = \cos(y) = \frac{\partial}{\partial x}[x\cos(y) + \cos(z)] = \frac{\partial v_y}{\partial x}$$

$$\frac{\partial v_x}{\partial z} = \frac{\partial}{\partial z}[\sin(y)] = 0 = \frac{\partial}{\partial x}[-y\sin(z)] = \frac{\partial v_z}{\partial x}$$

$$\frac{\partial v_y}{\partial z} = \frac{\partial}{\partial z}[x\cos(y) + \cos(z)] = -\sin(z) = \frac{\partial}{\partial y}[-y\sin(z)] = \frac{\partial v_z}{\partial y}$$

Hence $\vec{v} = \vec{\nabla} f$ is conservative.

Compute Potential:

$$f(x,y,z) = \int \frac{\partial f}{\partial x}\,dx = \int \sin(y)\,dx = x\sin(y) + C(y,z)$$

so $-y\sin(z) = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[x\sin(y) + C(y,z)] = C_z(y,z)$

and $C(y,z) = \int C_z(y,z)\,dz = \int -y\sin(z)\,dz = y\cos(z) + D(y)$

$\therefore f(x,y,z) = x\sin(y) + y\cos(z) + D(y)$

$\therefore x\cos(y) + \cos(z) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x\sin(y) + y\cos(z) + D(y)]$

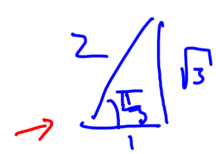
$= x\cos(y) + \cos(z) + D'(y)$

$\therefore D'(y) = 0 \quad \therefore D(y) = E$

Hence $f(x,y,z) = x\sin(y) + y\cos(z)$ is a potential.

Apply FTLI: $\vec{r}(t) = \langle \sin(t), t, 2t \rangle \quad 0 \le t \le \frac{\pi}{3}$

$$\int_C \vec{v} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

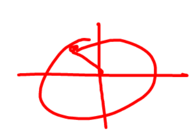


$= f\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}, \frac{2\pi}{3}\right) - f(0,0,0)$

$\vec{r}(0) = \langle 0, 0, 0 \rangle$

$\vec{r}\left(\frac{\pi}{3}\right) = \left\langle \frac{\sqrt{3}}{2}, \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle$

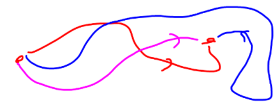
$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\pi}{3} \cdot -\frac{1}{2} - 0$



$= \frac{3}{4} - \frac{\pi}{6} = \frac{9 - 2\pi}{12}$

1/6

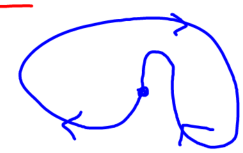
INDEPENDENCE OF PATH



Prop: If $\vec{v}$ is a conservative vector field then $\int_C \vec{v} \cdot d\vec{r}$ depends only on the endpoints of C (with orientation). Thus $\int_C \vec{v} \cdot d\vec{r}$ is independent of the path C.

pf: The Fundamental Theorem of Line Integrals! ///

Cor: If the path C is closed (i.e. has the same start and end point), then $\int_C \vec{v} \cdot d\vec{r} = 0$ for every conservative vector field $\vec{v}$.



$f(\vec{r}(b)) = f(\vec{r}(a))$

In a similar vein, we can prove the following:

Prop: Line integral $\int_C \vec{v} \cdot d\vec{r}$ is independent of path in $D \subseteq \mathbb{R}^n$ when $\int_C \vec{v} \cdot d\vec{r} = 0$ for all closed paths C in D.



Finally, the following is the result we're after ☺.

Prop: If $\vec{v}$ is a vector field continuous on an open disk $D \subseteq \mathbb{R}^n$ and $\int_C \vec{v} \cdot d\vec{r}$ is independant of path in D, then $\vec{v}$ is a conservative vector field.

pf: Pick a point $\vec{a} \in D$. Define $f(\vec{x}) = \int_{\vec{a}}^{\vec{x}} \vec{v} \cdot d\vec{r}$.
Then $f(\vec{x})$ is a potential for $\vec{v}$. ///

Ex: Let $\vec{v}(x,y) = \frac{1}{x^2+y^2}\langle -y, x\rangle$. Show $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. ✱

Then show $\int_C \vec{v}\cdot d\vec{r}$ is not independent of path.

Why does/doesn't this contradict the theorem above?

Come up with two paths C_1 and C_2 w/ the same endpoints s.t. $\int_{C_1} \vec{v}\cdot d\vec{r} \neq \int_{C_2} \vec{v}\cdot d\vec{r}$...

↳ Alternatively find a closed path C so that

✱ $\int_C \vec{v}\cdot d\vec{r} \neq 0$...